

Asymptotic behavior of solutions of an evolution equation for bidirectional surface waves in a convecting fluid

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Abstract

We consider the Cauchy problem for an evolution equation modeling bidirectional surface waves in a convecting fluid. Under small condition on the initial value, the existence and asymptotic behavior of global solutions in some time weighted spaces are established by the contraction mapping principle.

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1. Introduction

The Kuramoto-Sivashinsky (KS) equation

$$u_t + \gamma u_{xxxx} + \alpha u_{xx} + uu_x = 0 \quad (1.1)$$

is a well-known model of one-dimensional turbulence derived in various physical contexts such as chemical-reaction waves, propagation of combustion fronts in gases, surface waves in a film of a viscous liquid flowing along an inclined plane, patterns in thermal convection, rapid solidification (see e.g. [14, 21, 31]), where α and γ are constant coefficients accounting for the long-wave instability (gain) and short-wave dissipation, respectively. By combining the dispersive effects of the KdV equation and the dissipative effects of the KS equation, the Kuramoto-Sivashinsky-Korteweg-de Vries (KS-KdV) equation

$$u_t + u_{xxx} + \gamma u_{xxxx} + \alpha u_{xx} + uu_x = 0 \quad (1.2)$$

appears; which was first introduced by Benney [4]. This equation finds various applications in the study of unstable drift waves in plasmas [10], fluid flow along an inclined plane [4, 22] convection in fluids with a free surface [1–3, 13] the Eckhaus instability of traveling waves [15], in solar dynamo wave [18], hydrodynamics and other fields [9, 11, 20].

The derivation of this equation in the physical situations mentioned above involves the assumption of unidirectional waves. The assumption of unidirectional waves for surface waves was removed in [16, 19] and a modified Boussinesq system of equations was derived. One of these type of equation is the following dissipative Boussinesq equation:

$$u_{tt} - \Delta u + \Delta^2 u + \alpha \Delta u_t + \gamma \Delta^2 u_t = \Delta(\beta f(u_t) + g(u)). \quad (1.3)$$

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Here $u = u(x, t)$ is the unknown function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t > 0$ and $\beta > 0$ and $\alpha \in \mathbb{R}$ are constants. The term u_t represents a frictional function dissipation, and the nonlinear term $f(v)$ and $g(v)$ are smooth functions of v under considerations and satisfies $f(v) = O(|v|^2)$ and $g(v) = O(|v|^2)$ for $v \rightarrow 0$. Equation (1.3) arises in the study of the stability of one-dimensional periodic patterns in systems with Galilean invariance and also the oscillations of elastic beams [6–8]. Ignoring the dissipation, (1.3) turns into the classical Boussinesq equation

$$u_{tt} - \Delta u \pm \Delta^2 u = \Delta(u^2); \quad (1.4)$$

appeared not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of the nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods and in the continuum limit of lattice dynamics or coupled electrical circuit. When $\gamma = \beta = 0$, the existence, uniqueness and long-time asymptotic of solutions to the Cauchy problem and the initial boundary value problem of equation (1.3) has been studied by several authors, see for instance [12, 17, 23–30] and references therein.

In this paper we study the asymptotic behavior of solutions of the Cauchy problem associated to (1.3) with the initial values

$$u(0) = u_0(x), \quad u_t(0) = u_1(x). \quad (1.5)$$

The article is organized as follows. In Section 2 we obtain the solution formula of (1.3) and study the decay property of the solution operators appearing in the solution formula. Then, in Section 3, we discuss the linear problem and show the decay estimates of the solutions in L^1 . We prove global existence and asymptotic behavior of solutions for the Cauchy problem (1.3) and (1.5) in L^2 in Section 4.

Throughout this paper we assume $\gamma = 1 \leq -\alpha$.

2. Decay property of the linear part

The aim of this section is to derive the solution formula for the problem (1.3) and (1.5). First of all, we investigate the linear equation of (1.3).

$$u_{tt} - \Delta u + \Delta^2 u + \alpha \Delta u_t + \Delta^2 u_t = 0 \quad (2.1)$$

with the initial data (1.5).

By applying the Fourier transform to (2.1) we have

$$\hat{u}_{tt} + (|\xi|^4 - \alpha|\xi|^2)\hat{u}_t + (|\xi|^2 + |\xi|^4)\hat{u} = 0. \quad (2.2)$$

The corresponding initial values are given as

$$t = 0: \quad \hat{u} = \hat{u}_0(\xi), \quad \hat{u}_t = \hat{u}_1(\xi). \quad (2.3)$$

The characteristic equation of (2.2) is

$$\lambda^2 + (|\xi|^4 - \alpha|\xi|^2)\lambda + (|\xi|^2 + |\xi|^4) = 0. \quad (2.4)$$

Let $\lambda = \lambda_{\pm}(\xi)$ be the corresponding eigenvalues, i.e

$$\lambda_{\pm}(\xi) = \frac{(\alpha|\xi|^2 - |\xi|^4) \pm \sqrt{(|\xi|^4 - \alpha|\xi|^2)^2 - 4(|\xi|^2 + |\xi|^4)}}{2}. \quad (2.5)$$

The solution to the problem (2.2) and (2.3) is given in the form

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi) + \hat{H}(\xi, t)\hat{u}_0(\xi), \quad (2.6)$$

where

$$\hat{G}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} (e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}) \quad (2.7)$$

and

$$\hat{H}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} (\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}). \quad (2.8)$$

Let

$$G(x, t) = F^{-1}[\hat{G}(\xi, t)](x) \quad (2.9)$$

and

$$H(x, t) = F^{-1}[\hat{H}(\xi, t)](x), \quad (2.10)$$

where F^{-1} denotes the inverse Fourier transform. With applying F^{-1} to (2.6), we obtain

$$u(t) = G(t) * u_1 + H(t) * u_0. \quad (2.11)$$

By the Duhamel principle, we obtain the solution formula to (1.3) and (1.5)

$$u(t) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t - \tau) * \Delta(f(u(\tau)) + \beta g(u_t))(\tau) d\tau. \quad (2.12)$$

Now we study the decay property of the linear equation (1.3). Our aim is to prove the following decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in (2.11)

Lemma 2.1. *The solution of (2.2) and (2.3) satisfies*

$$|\xi|^2(1 + |\xi|^2)|\hat{u}(\xi, t)|^2 + |\hat{u}_t(\xi, t)|^2 \leq Ce^{-c\omega(\xi)t}(|\xi|^2(1 + |\xi|^2)|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) \quad (2.13)$$

for $\xi \in R^n$ and $t \geq 0$, where $\omega(\xi) = |\xi|^2$.

Proof. By multiplying (2.2) by $\bar{\hat{u}}_t$ and taking the real part, we deduce that

$$\frac{1}{2} \frac{d}{dt} (|\hat{u}_t|^2 + (|\xi|^2 + |\xi|^4)|\hat{u}|^2) + (|\xi|^4 - \alpha|\xi|^2)|\hat{u}_t|^2 = 0 \quad (2.14)$$

Multiplying (2.2) by $\bar{\hat{u}}$ and take the real part yields

$$\frac{1}{2} \frac{d}{dt} ((|\xi|^4 - \alpha|\xi|^2)|\hat{u}|^2 + 2\operatorname{Re}(\hat{u}_t \bar{\hat{u}})) + (|\xi|^2 + |\xi|^4)|\hat{u}|^2 - |\hat{u}_t|^2 = 0 \quad (2.15)$$

Multiplying both sides of (2.14) and (2.15) by $(1 + |\xi|^2)$ and $|\xi|^2$ respectively, summing up the products yields

$$\frac{d}{dt} E + F = 0, \quad (2.16)$$

where

$$E = (1 + |\xi|^2)|\hat{u}_t|^2 + \{(1 + |\xi|^2)(|\xi|^2 + |\xi|^4) + |\xi|^2(|\xi|^4 - \alpha|\xi|^2)\}|\hat{u}|^2 + 2|\xi|^2 \operatorname{Re}(\hat{u}_t \bar{\hat{u}})$$

and

$$F = \{2(1 + |\xi|^2)(|\xi|^4 - \alpha|\xi|^2) - 2|\xi|^2\} |\hat{u}_t|^2 + 2|\xi|^2(|\xi|^2 + |\xi|^4).$$

It is easy to see that

$$C(1 + |\xi|^2)E_0 \leq E \leq C(1 + |\xi|^2)E_0, \quad (2.17)$$

where

$$E_0 = |\hat{u}_t|^2 + |\xi|^2(1 + |\xi|^2)|\hat{u}|^2.$$

Noting that $F \geq |\xi|^2 E_0$ and with (2.17), we obtain

$$F \geq c \omega(\xi) E, \quad (2.18)$$

where

$$\omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}.$$

Using (2.16) and (2.18), we get

$$\frac{d}{dt} E + c \omega(\xi) E \leq 0.$$

Thus

$$E(\xi, t) \leq e^{-c \omega(\xi) t} E(\xi, 0),$$

which together with (2.17) proves the desired estimate (2.13). \square

Lemma 2.2. *Assume that $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ are fundamental solutions of (2.1) in the Fourier space, which are given explicitly in (2.7) and (2.8). Then we have the pointwise estimates*

$$|\xi|^2 (1 + |\xi|^2) |\hat{G}(\xi, t)|^2 + |\hat{G}_t(\xi, t)|^2 \leq C e^{-c \omega(\xi) t} \quad (2.19)$$

and

$$|\xi|^2 (1 + |\xi|^2) |\hat{H}(\xi, t)|^2 + |\hat{H}_t(\xi, t)|^2 \leq C |\xi|^2 (1 + |\xi|^2) e^{-c \omega(\xi) t}, \quad (2.20)$$

for $\xi \in \mathbb{R}^n$ and $t \geq 0$, where $\omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}$.

Proof. If $\hat{u}_0(\xi) = 0$, then from (2.6) we get

$$\hat{u}(\xi, t) = \hat{G}(\xi, t) \hat{u}_1(\xi), \quad \hat{u}_t(\xi, t) = \hat{G}_t(\xi, t) \hat{u}_1(\xi).$$

Substituting the equalities into (2.13) with $\hat{u}_0(\xi) = 0$ we obtain (2.19). In what follows, we consider $\hat{u}_1(\xi) = 0$. We have from (2.6) that

$$\hat{u}(\xi, t) = \hat{H}(\xi, t) \hat{u}_0(\xi), \quad \hat{u}_t(\xi, t) = \hat{H}_t(\xi, t) \hat{u}_0(\xi).$$

Substituting the equalities into (2.13) with $\hat{u}_1(\xi) = 0$, we obtain (2.20), which together with (2.19), we have completed the proof of the lemma. \square

Lemma 2.3. *Let l, k, j be nonnegative integers and assume that $1 \leq p \leq 2$. Then we have*

$$\begin{aligned} \|\partial_x^k G(t) * \phi\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}} \|\partial_x^j \phi\|_{\dot{W}^{-1,p}} \\ &\quad + C e^{-ct} \|\partial_x^{k+l-2} \phi\|_{L^2}, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \|\partial_x^k H(t) * \psi\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}} \|\partial_x^j \psi\|_{L^p} \\ &\quad + C e^{-ct} \|\partial_x^{k+l} \phi\|_{L^2}, \end{aligned} \quad (2.22)$$

for $0 \leq j \leq k$, where $k+l-2 \geq 0$ in (2.21). Similarly, we have

$$\begin{aligned} \|\partial_x^k G_t(t) * \phi\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}} \|\partial_x^j \phi\|_{\dot{W}^{-1,p}} \\ &\quad + C e^{-ct} \|\partial_x^{k+l} \phi\|_{L^2}, \end{aligned} \quad (2.23)$$

$$\begin{aligned} \|\partial_x^k H_t(t) * \psi\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}} \|\partial_x^j \psi\|_{L^p} \\ &\quad + Ce^{-ct} \|\partial_x^{k+l+2} \phi\|_{L^2}, \end{aligned} \quad (2.24)$$

for $0 \leq j \leq k+1$.

Proof. We only give a proof of (2.21). We apply the Plancherel theorem and use the pointwise estimate for \hat{G} in (2.19). This gives

$$\begin{aligned} \|\partial_x^k G_t(t) * \phi\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 d\xi \\ &= \int_{|\xi| \leq 1} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 d\xi + \int_{|\xi| \geq 1} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 d\xi \\ &\leq \int_{|\xi| \leq 1} |\xi|^{2k-2} e^{-c|\xi|^2 t} |\hat{\phi}(\xi)|^2 d\xi \\ &\quad + C \int_{|\xi| \geq 1} e^{-c\omega(\xi)t} |\xi|^{2k} (|\xi|^2(1+|\xi|^2))^{-1} |\hat{\phi}(\xi)|^2 d\xi \\ &\leq C \|\xi|^{j-1} \hat{\phi}(\xi)\|_{L^p}^2 \left(\int_{|\xi| \leq 1} |\xi|^{2(k-j)q} e^{-cq|\xi|^2 t} d\xi \right)^{\frac{1}{q}} \\ &\quad + Ce^{-ct} \int_{|\xi| \geq 1} |\xi|^{2k-4} |\hat{\phi}(\xi)|^2 d\xi \\ &\leq C \|\xi|^{j-1} \hat{\phi}(\xi)\|_{L^p}^2 (\|\xi|^{2(k-j)} e^{-c|\xi|^2 t}\|_{L^q} \\ &\quad + Ce^{-ct} \int_{|\xi| \geq 1} |\xi|^{2(k+l-2)} |\hat{\phi}(\xi)|^2 d\xi, \end{aligned}$$

where we used Holder inequality with $\frac{1}{q} + \frac{2}{p} = 1$, $\frac{1}{p} + \frac{1}{p} = 1$. With a straight computation, we obtain

$$\|\xi|^{2(k-j)} e^{-c|\xi|^2 t}\|_{L^q(|\xi| \leq 1)} \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-(k-j)}.$$

It follows from the Hausdorff-Young inequality that

$$\|\xi|^{j-1} \hat{\phi}(\xi)\|_{L^p} \leq \|\partial_x^j \phi\|_{\dot{W}^{-1,p}}$$

Combining the above three inequalities yields (2.21). Similarly, we can prove (2.22)-(2.24). Thus the lemma is proved. \square

Immediately we have from previous lemma the following corollary.

Corollary 2.1. *Let $1 \leq p \leq 2$, and let k, j and l be nonnegative integers. Also, assume that $G(x, t)$ and $H(x, t)$ be the fundamental solution of (2.1) which are given in (2.7) and (2.8), respectively. Then we have*

$$\begin{aligned} \|\partial_x^k G(t) * \Delta g\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}} \|\partial_x^j g\|_{L^p} \\ &\quad + Ce^{-ct} \|\partial_x^{k+l} g\|_{L^2}, \end{aligned} \quad (2.25)$$

for $0 \leq k \leq j+1$. It also for $0 \leq k \leq j+2$ holds that

$$\|\partial_x^k G_t(t) * \Delta g\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+2-j}{2}} \|\partial_x^j g\|_{L^p} + Ce^{-ct} \|\partial_x^{k+l+2} g\|_{L^2}. \quad (2.26)$$

3. Global existence and asymptotic behavior of solutions for L^1

The aim of this section is to prove the existence and asymptotic behavior of solutions to (1.3) and (1.5) with L^1 data. We first state the following lemma, which comes from [32].

Lemma 3.1. *Assume that $f = f(v)$ is smooth function, where $v = (v_1, \dots, v_n)$ is a vector function. Suppose that $f(v) = O(|v|^{1+\theta})$ ($\theta \geq 1$ is an integer) when $|v| \leq v_0$. Then, for the integer $m \geq 0$, if $v, w \in W^{m,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty} \leq v_0, \|w\|_{L^\infty} \leq v_0$, then $f(v) - f(w) \in W^{m,r}(\mathbb{R}^n)$. Furthermore, the following inequalities hold:*

$$\|\partial_x^m f(v)\|_{L^r} \leq C \|v\|_{L^p} \|\partial_x^m v\|_{L^q} \|v\|_{L^\infty}^{\theta-1} \quad (3.1)$$

and

$$\begin{aligned} \|\partial_x^m (f(v) - f(w))\|_{L^r} &\leq C \{ (\|\partial_x^m v\|_{L^q}) \|v - w\|_{L^p} + \\ &\quad (\|v\|_{L^p} + \|w\|_{L^p}) \|\partial_x^m (v - w)\|_{L^q} \} (\|v\|_{L^\infty} + \|w\|_{L^\infty})^{\theta-1}, \end{aligned} \quad (3.2)$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 \leq p, q, r \leq +\infty$.

Based on the decay estimates of solutions to the linear problem (2.1), we define the following solution space:

$$X = \{u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t \geq 0} \left\{ \sum_{k \leq s+2} (1+t)^{\frac{n}{4} + \frac{k}{2}} \|\partial_x^k u(t)\|_{L^2} + \sum_{k \leq s} (1+t)^{\frac{n}{4} + \frac{k}{2}} \|\partial_x^k u_t(t)\|_{L^2} \right\}.$$

For $R > 0$, we define

$$X_R = \{u \in X : \|u\|_X \leq R\}.$$

Theorem 3.1. *Let $n \geq 1, s \geq \max\{0, [\frac{n}{2}] - 1\}$. Suppose that $u_0 \in H^{s+2}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n) \cap \dot{W}^{-1,1}(\mathbb{R}^n)$ and $f(v), g(v)$ are smooth and satisfies $f(v) = O(v^2)$, $g(v) = O(v^2)$ for $v \rightarrow 0$. Put*

$$E_0 := \|u_0\|_{L^1} + \|u_1\|_{\dot{W}^{-1,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

If E_0 is suitably small, the Cauchy problem (1.3) and (1.5) has a unique global solution $u(x, t)$ satisfying

$$X = u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)).$$

Also, the solution satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \leq C E_0 (1+t)^{-\frac{n}{4} - \frac{k}{2}} \quad (3.3)$$

and

$$\|\partial_x^l u_t(t)\|_{L^2} \leq C E_0 (1+t)^{-\frac{n}{4} - \frac{l+1}{2}} \quad (3.4)$$

for $0 \leq k \leq s+2$ and $0 \leq l \leq s$.

Proof. The Gagliardo-Nirenberg inequality gives

$$\|u(t)\|_{L^\infty} \leq C \|\partial_x^{s_0} u\|_{L^2}^\theta \|u\|_{L^2}^{1-\theta} \leq C (1+t)^{-\frac{n}{2}} \|u\|_X \quad (3.5)$$

where $s_0 = \frac{n}{2} + 1$, $\theta = \frac{n}{2s_0}$; i.e, $s \geq [\frac{n}{2}] - 1$. We define

$$\Phi(u) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t - \tau) * \Delta(f(u) - \beta g(u_t))(\tau) d\tau.$$

We apply ∂_x^k to Φ and take the L^2 norm. We obtain

$$\begin{aligned} \|\partial_x^k \Phi(u)\|_{L^2} &\leq \|\partial_x^k G(t) * u_1\|_{L^2} + \|\partial_x^k H(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^k G(t - \tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &:= I_1 + I_2 + J \end{aligned} \quad (3.6)$$

First, we estimate I_1 . We apply (2.21) with $p = 1$, $j = 0$, $l = 0$ and get

$$\begin{aligned} I_1 &\leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|u_1\|_{\dot{W}^{-1,1}} + Ce^{-ct} \|\partial_x^{(k-2)+} u_1\|_{L^2} \\ &\leq CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \end{aligned} \quad (3.7)$$

where $(k-2)_+ = \max\{k-2, 0\}$.

For the term I_2 , we apply (2.22) with $p = 1$, $j = 0$ and $l = 0$. This yields

$$I_2 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|u_0\|_{L^1} + Ce^{-ct} \|\partial_x^k u_0\|_{L^2} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}. \quad (3.8)$$

Next, we estimate J . Let

$$\begin{aligned} J &= \int_0^t G(t - \tau) * \Delta(f(u) - \beta g(u_t))(\tau) d\tau \\ &= \int_0^{t/2} G(t - \tau) * \Delta(f(u) - \beta g(u_t))(\tau) d\tau \\ &\quad + \int_{t/2}^t G(t - \tau) * \Delta(f(u) - \beta g(u_t))(\tau) d\tau \\ &=: J_1 + J_2 \end{aligned}$$

For the term J_1 , using (2.25) with $p = 1$, $j = 0$ and $l = 0$, we have

$$\begin{aligned} J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \|f(u)(\tau) - \beta g(u_t)(\tau)\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^k (f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: J_{11} + J_{12} \end{aligned} \quad (3.9)$$

Note that by lemma (3.1) we have

$$\begin{aligned} \|f(u)\|_{L^1} &\leq C \|u\|_{L^2}^2 \leq CR^2(1+\tau)^{-\frac{n}{2}} \\ \|g(u_t)\|_{L^1} &\leq C \|u_t\|_{L^2}^2 \leq CR^2(1+\tau)^{-\frac{n}{2}}. \end{aligned}$$

Therefore we have

$$\begin{aligned}
J_{11} &\leq CR^2 \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} (1+\tau)^{-\frac{n}{2}} d\tau \\
&\leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \int_0^{t/2} (1+\tau)^{\frac{n}{2}} d\tau \\
&\leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}} \eta(t),
\end{aligned}$$

where

$$\eta(t) = \begin{cases} 1, & n = 1 \\ (1+t)^{-\frac{1}{2}} \ln(2+t), & n = 2 \\ (1+t)^{-\frac{1}{2}}, & n \geq 3. \end{cases} \quad (3.10)$$

We use (3.1) and obtain

$$\|\partial_x^k(f(u)(\tau) - \beta g(u_t(\tau)))\|_{L^2} \leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}} \quad (3.11)$$

Consequently, we get

$$J_{12} \leq CR^2 \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}} d\tau \leq CR^2 e^{-ct}.$$

Finally, we estimate the term J_2 on the time interval $[t/2, t]$. Applying (2.25) with $p = 2, j = k, l = 0$ and using (3.11), we can estimate term J_2 as

$$\begin{aligned}
J_2 &\leq C \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^k(f(u) - \beta g(u_t)(\tau))\|_{L^2} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^k(f(u) - \beta g(u_t)(\tau))\|_{L^2} d\tau \\
&\leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n-1}{2}}.
\end{aligned} \quad (3.12)$$

Thus we have shown that

$$J \leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}} \eta(t).$$

Substituting all this estimate into (3.6), we have

$$(1+t)^{\frac{n}{4}+\frac{k}{2}} \|\partial_x^k \Phi(u)\| \leq CE_0 + CR^2, \quad (3.13)$$

for $0 \leq k \leq s+2$. It follows from that (3.6)

$$\begin{aligned}
\Phi(u)_t &= G_t(t) * u_1 + H_t(t) * u_0 \\
&\quad + \int_0^t G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)_{L^2} d\tau.
\end{aligned} \quad (3.14)$$

We use ∂_x^k to $\Phi(u)_t$ and take L^2 norm. This yields

$$\begin{aligned}
\|\partial_x^k \Phi(u)_t\|_{L^2} &\leq \|\partial_x^k G_t(t) * u_1\|_{L^2} + \|\partial_x^k H_t(t) * u_0\|_{L^2} \\
&\quad + C \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\
&=: \dot{I}_1 + \dot{I}_2 + \dot{J},
\end{aligned} \quad (3.15)$$

for $0 \leq k \leq s$. For the term \dot{I}_1 , we apply (2.23) with $p = 1, j = 0$ and $l = 0$ and obtain

$$\dot{I}_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \|u_1\|_{\dot{W}^{-1,1}} + Ce^{-ct} \|\partial_x^k u_1\|_{L^2} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}.$$

Also, for the term \dot{I}_2 , we apply (2.24) with $p = 1, j = 0$ and $l = 0$ and get

$$\dot{I}_2 \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \|u_0\|_{L^1} + Ce^{-ct} \|\partial_x^{k+2} u_0\|_{L^2} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}.$$

To estimate the nonlinear term \dot{J} , we divide as $\dot{J} = \dot{J}_1 + \dot{J}_2$, where \dot{J}_1 and \dot{J}_2 correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. By applying (2.26) with $p = 1, j = 0$ and $l = 0$, we have

$$\begin{aligned} \dot{J}_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|f(u)(\tau) - \beta g(u_t)(\tau)\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: \dot{J}_{11} + \dot{J}_{12}. \end{aligned}$$

By (3.1), we obtain

$$\|(f(u) - \beta g(u_t))(\tau)\|_{L^1} \leq CR^2(1+\tau)^{-\frac{n}{2}}.$$

Therefor we get

$$\begin{aligned} \dot{J}_{11} &\leq CR^2 \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} (1+\tau)^{-\frac{n}{2}} d\tau \\ &\leq CR^2(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \eta(t). \end{aligned}$$

Similarly as before, we can estimate \dot{J}_{12} and obtain $\dot{J}_{12} \leq CR^2 e^{-ct}$. Finally, we estimate the term \dot{J}_2 by using (2.26) with $p = 2, j = k+2, l = 0$ and get

$$\begin{aligned} \dot{J}_2 &\leq C \int_{t/2}^t \|\partial_x^{k+2}(f(u) - \beta g(u_t)(\tau))\|_{L^2} d\tau \\ &\quad + C \int_{t/2}^t -\frac{n}{4} - \frac{k+1}{2} - \frac{n-1}{2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - \beta g(u_t)(\tau))\|_{L^2} d\tau \\ &\leq CR^2 \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{k+1}{2}-\frac{n}{2}} d\tau \\ &\leq CR^2(1+t)^{-\frac{n}{4}-\frac{k+1}{2}-\frac{n-1}{2}}. \end{aligned}$$

Consequently we have that

$$\dot{J} \leq CR^2(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \eta(t).$$

The above inequality implies

$$(1+t)^{\frac{n}{4}+\frac{k+1}{2}} \|\partial_x^k \Phi(u)_t\|_{L^2} \leq CE_0 + CR^2. \quad (3.16)$$

Combining (3.16) and (3.13) and taking E_0 and R suitably small, we obtain $\|\Phi(u)\|_X \leq R$. For $u, \tilde{u} \in X_R$, (3.6) gives

$$\begin{aligned}
\|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} &= \int_0^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u})) \\
&\quad - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&= \int_0^{t/2} \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u})) \\
&\quad - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&\quad + \int_{t/2}^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u})) \\
&\quad - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&=: J_1 + J_2
\end{aligned}$$

For the term J_1 , we apply (2.25) with $p = 1$, $j = 0$ and $l = 0$, we arrive at

$$\begin{aligned}
J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \|(f(u) - f(\tilde{u}))(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\
&\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)))(\tau)\|_{L^2} d\tau \\
&=: J_{11} + J_{12}
\end{aligned} \tag{3.17}$$

By (3.2), we can estimate J_{11} as

$$\begin{aligned}
J_{11} &\leq CR \|u - \tilde{u}\|_X \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} (1+\tau)^{-\frac{n}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{n}{4}-\frac{k}{2}} (1+t)^{-\frac{n}{4}-\frac{k}{2}} \eta(t),
\end{aligned}$$

where η be defined in (3.10). It follows from the Gagliardo-Nirenberg inequality and (3.2) that

$$\begin{aligned}
J_{12} &\leq \int_0^{t/2} e^{-c(t-\tau)} \left[(\|\partial_x^k u\|_{L^2} + \|\partial_x^k \tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} \right. \\
&\quad + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^k(u - \tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^\infty} \\
&\quad \left. + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^k(u_t - \tilde{u}_t)\|_{L^2} \right] d\tau \\
&\leq CR \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}} \|u - \tilde{u}\|_X d\tau \\
&\leq CR \|u - \tilde{u}\|_X e^{-ct}.
\end{aligned}$$

Finally, we estimate term J_2 on the time $[t/2, t]$. Applying (2.25) with $p = 2$, $j = k$, $l = 0$ and using (3.2), we obtain

$$\begin{aligned}
J_2 &\leq C \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&\leq CR \|u - \tilde{u}\|_X \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} (1+t)^{-\frac{n}{4} - \frac{k}{2} - \frac{n}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{n}{4} - \frac{k}{2} - \frac{n-1}{2}};
\end{aligned}$$

which implies

$$(1+t)^{\frac{n}{4} + \frac{k}{2}} \|\partial_x^k \Phi(u) - \Phi(\tilde{u})\|_{L^2} \leq CR \|u - \tilde{u}\|_X. \quad (3.18)$$

Similarly, for $0 \leq k \leq s$ and $u, \tilde{u} \in X$ from (3.2), (2.26), we deduce that

$$\begin{aligned}
\|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))_t\|_{L^2} &= \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&= \int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&\quad + \int_{t/2}^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&=: J_1 + J_2
\end{aligned}$$

For the term J_1 , we use (2.26) with $p = 1$, $j = 0$ and $l = 2$. We have

$$\begin{aligned}
J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4} - \frac{k+2}{2}} \|f(u) - f(\tilde{u})(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\
&\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}_t) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&=: J_{11} + J_{12}.
\end{aligned}$$

By (3.2), we have

$$\begin{aligned}
J_{11} &\leq \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4} - \frac{k+2}{2}} (\|u\|_{L^2} + \|\tilde{u}_{L^2}\|) \|u - \tilde{u}\|_{L^2} \\
&\quad + (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) (\|u_t - \tilde{u}_t\|_{L^2}) d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{n}{4} - \frac{k+1}{2}} \int_0^{t/2} (1+t-\tau)^{-\frac{n}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{n}{4} - \frac{k+1}{2}} \eta(t),
\end{aligned}$$

where η be defined in (3.10). Also, the term J_{12} is estimated similarly as before and we can estimate the term J_{12} as

$$J_{12} \leq CR \|u - \tilde{u}\|_X e^{-ct}.$$

By applying (2.26) with $p = 2$, $j = k + 2$, $l = 0$ and (3.1), we obtain

$$\begin{aligned}
J_2 &\leq C \int_{t/2}^t \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\
&\leq \int_{t/2}^t (\|\partial_x^{k+2}u\|_{L^2} + \|\partial_x^{k+2}\tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u})\|_{L^2} \\
&\quad + ((\|\partial_x^{k+2}u_t\|_{L^2} + \|\partial_x^{k+2}\tilde{u}_t\|_{L^2}) \|u - \tilde{u}_t\|_{L^\infty} + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u}_t)\|_{L^2}) d\tau \\
&\leq CR \|u - \tilde{u}\|_X \int_{t/2}^t (1 + \tau)^{-\frac{n}{4} - \frac{k+2}{2} - \frac{n}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1 + t)^{-\frac{n}{4} - \frac{k+1}{2} - \frac{n-1}{2}} \\
&\leq CR \|u - \tilde{u}\|_X (1 + t)^{-\frac{n}{4} - \frac{k+1}{2}}
\end{aligned}$$

Substituting all these estimates together with the previous estimate and taking R suitably small, yields

$$\|\Phi(u) - \Phi(\tilde{u})\|_X \leq \frac{1}{2} \|u - \tilde{u}\|_X. \quad (3.19)$$

From (3.19), we deduce that Φ is strictly contracting mapping. Then there exists a fixed point $u \in X_R$ of the mapping Φ , which is a solution to (1.3), (1.5). The proof of the theorem is now complete. \square

The proof of the previous theorem shows that when $n \geq 2$, the solution u to the integral equation (2.10) is asymptotic to the linear solution $u_L(t)$ given by the formula $u_L(t) = G(t) * u_1 + H(t) * u_0$ as $t \rightarrow \infty$. This result is stated as follows.

Lemma 3.2. *Let $n \geq 2$ and assume the same conditions of Theorem (3.1). Then the solution u of the problem (1.3), (1.5) which is constructed in theorem (3.1), can be approximated by the solution u_L to the linearized problem (2.1), (2.2) as $t \rightarrow \infty$. More precisely, we have*

$$\|\partial_x^k(u - u_L)(t)\|_{L^2} \leq CE_0^2(1 + t)^{-\frac{n}{4} - \frac{k}{2}} \eta(t),$$

$$\|\partial_x^k(u - u_L)_t(t)\|_{L^2} \leq CE_0^2(1 + t)^{-\frac{n}{4} - \frac{k+1}{2}} \eta(t),$$

for $0 \leq k \leq s + 2$ and $0 \leq k \leq s$, respectively, where $u_L(t) := G(t) * u_1 + H(t) * u_0$ is the linear solution and $\eta(t)$ is defined in (3.10).

4. Decay estimates of solutions for L^2

In the previous section, we have proved global existence and asymptotic behavior of solutions to the Cauchy problem (1.3), (1.5) with L^1 data.

In this section, we prove a similar decay estimate of solution with L^2 data for $n \geq 2$. Based on the decay estimates of solutions to the linear problem (2.1), (2.2), we define the following solution space:

$$X = \{u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t \geq 0} \left\{ \sum_{k \leq s+2} (1 + t)^{\frac{k}{2}} \|\partial_x^k u(t)\|_{L^2} + \sum_{k \leq s} (1 + t)^{\frac{k}{2}} \|\partial_x^k u_t(t)\|_{L^2} \right\}.$$

For $R > 0$, we define

$$X_R = \{u \in X : \|u\|_X \leq R\}.$$

Note that from the Gagliardo-Nirenberg inequality for $u \in X_R$, we have

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{4}}. \quad (4.1)$$

Theorem 4.1. *Suppose that $u_0 \in H^{s+2}$, $u_1 \in H^s(\mathbb{R}^n) \cap \dot{W}^{-1,2}(\mathbb{R}^n)$, such that $n \geq 1$, $s \geq \max\{0, [\frac{n}{2}] - 1\}$, and $f(v), g(v)$ are smooth and satisfies $f(v) = O(v^2), g(v) = O(v^2)$ for $v \rightarrow 0$. Let*

$$E_1 := \|u_0\|_{L^2} + \|u_1\|_{\dot{W}^{-1,2}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

If E_0 is suitably small, the Cauchy problem (1.3) and (1.5) has a unique global solution $u(x, t)$ satisfying

$$X = u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)).$$

The solution u also satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{k}{2}} \quad (4.2)$$

and

$$\|\partial_x^h u_t(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{h+1}{2}}. \quad (4.3)$$

for $0 \leq k \leq s+2$ and $0 \leq h \leq s$.

Proof. Let the mapping Φ be defined in (3.6). Applying ∂_x^k to Φ and take L^2 norm. We have

$$\begin{aligned} \|\partial_x^k \Phi(u)\|_{L^2} &\leq \|\partial_x^k G(t) * u_1\|_{L^2} + \|\partial_x^k H(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &:= I_1 + I_2 + J. \end{aligned} \quad (4.4)$$

We use (2.21) with $p=2, j=l=0$ and get

$$\begin{aligned} I_1 &\leq C(1+t)^{-\frac{k}{2}} \|u_1\|_{\dot{W}^{-1,2}} + Ce^{-ct} \|\partial_x^{(k-2)+} u_1\|_{L^2} \\ &\leq CE_1(1+t)^{-\frac{k}{2}}, \end{aligned} \quad (4.5)$$

where $(k-2)_+ = \max\{k-2, 0\}$. By applying (2.22) with $p=2, j=l=0$, we get

$$\begin{aligned} I_2 &\leq C(1+t)^{-\frac{k}{2}} \|u_0\|_{L^2} + Ce^{-ct} \|\partial_x^k u_0\|_{H^{s+2}} \\ &\leq CE_1(1+t)^{-\frac{k}{2}}. \end{aligned} \quad (4.6)$$

To estimate the nonlinear J , as in the pervious section, we divide as $J = J_1 + J_2$ where J_1 and J_2 correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. For the term J_1 , we use (2.25) with $p=1, j=l=0$ and deduce that

$$\begin{aligned} J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \|f(u)(\tau) - \beta g(u_t)(\tau)\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^k (f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: J_{11} + J_{12} \end{aligned} \quad (4.7)$$

By (3.1), we have $\|f(u) - \beta g(u_t)(\tau)\|_{L^1} \leq CR^2$. Thus we can estimate the J_{11} as

$$\begin{aligned} J_{11} &\leq CR^2 \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} (1+\tau)^{-\frac{n}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k}{2}} \int_0^{t/2} (1+\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k}{2}}. \end{aligned}$$

By applying (3.1) and Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} \|\partial_x^k f(u)\|_{L^2} &\leq C\|u\|_{L^\infty} \|\partial_x^k u\|_{L^2} \leq C(1+t)^{\frac{n}{4}-\frac{k}{2}} R^2 \\ \|\partial_x^k g(u_t)\|_{L^2} &\leq C\|u_t\|_{L^\infty} \|\partial_x^k u_t\|_{L^2} \leq C(1+t)^{\frac{n}{4}-\frac{k}{2}} R^2 \end{aligned} \quad (4.8)$$

Thus we have

$$J_{12} \leq CR^2 \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}} d\tau \leq CR^2 e^{-ct}.$$

It follows from (2.25) with $p=1, j=k$ and $l=2$ that

$$\begin{aligned} J_2 &\leq C \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} \|\partial_x^k (f(u) - \beta g(u_t))(\tau)\|_{L^1} d\tau \\ &\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2} (f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: J_{21} + J_{22} \end{aligned}$$

We have using (3.2) that

$$\begin{aligned} J_{21} &\leq C \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} (\|u\|_{L^2} \|\partial_x^k u\|_{L^2} + \|u_t\|_{L^2} \|\partial_x^k u_t\|_{L^2}) d\tau \\ &\leq CR^2 \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} (1+\tau)^{-\frac{k}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k}{2}} \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k}{2}} \end{aligned}$$

To estimate the term J_{22} , by (4.8), we have

$$\begin{aligned} J_{22} &\leq C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2} (f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &\leq CR^2 \int_{t/2}^t e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k}{2}}. \end{aligned}$$

The above inequality shows that

$$(1+t)^{\frac{k}{2}} \|\partial_x^k \Phi(u)\| \leq CE_1 + CR^2 \quad (4.9)$$

We deduce from (3.6) that

$$\Phi(u)_t = G_t(t) * u_1 + H_t(t) * u_0 + \int_0^t G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau) \|_{L^2} d\tau. \quad (4.10)$$

Applying ∂_x^k to $\Phi(u)_t$ and taking L^2 -norm we have

$$\begin{aligned} \|\partial_x^k \Phi(u)_t\|_{L^2} &\leq \|\partial_x^k G_t(t) * u_1\|_{L^2} + \|\partial_x^k H_t(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: \acute{I}_1 + \acute{I}_2 + \acute{J}, \end{aligned} \quad (4.11)$$

To estimate the term \acute{I}_1 , apply (2.23) with $p = 2, l = j = 0$. It yields

$$\acute{I}_1 \leq C(1+t)^{-\frac{k+1}{2}} \|u_1\|_{\dot{W}^{-1,2}} + Ce^{-ct} \|\partial_x^k u_1\|_{L^2} \leq CE_1(1+t)^{-\frac{k+1}{2}}.$$

Similarly, using (2.24) with $p = 2, j = l = 0$, we have

$$\acute{I}_2 \leq C(1+t)^{-\frac{k+1}{2}} \|u_0\|_{L^2} + Ce^{-ct} \|\partial_x^{k+2} u_0\|_{L^2} \leq CE_1(1+t)^{-\frac{k+1}{2}}.$$

To estimate the nonlinear term \acute{J} , let

$$\begin{aligned} \acute{J} &= C \int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &\quad + C \int_{t/2}^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &=: \acute{I}_1 + \acute{I}_2 + \acute{J}, \end{aligned}$$

Using (2.26) with $p = 1, j = l = 0$, it yields

$$\begin{aligned} \acute{J}_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|(f(u)(\tau) - \beta g(u_t))(\tau)\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - \beta g(u_t))(\tau)\|_{L^2} d\tau \\ &:= \acute{J}_{11} + \acute{J}_{12}. \end{aligned}$$

By (3.1), we obtain

$$\|(f(u) - \beta g(u_t))(\tau)\|_{L^1} \leq CR^2.$$

Thus we can estimate \acute{J}_{11} as

$$\begin{aligned} \acute{J}_{11} &\leq CR^2 \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ &\leq CR^2(1+t)^{-\frac{k+1}{2}} \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR^2(1+t)^{-\frac{k+1}{2}} \end{aligned}$$

For the term J'_{12} , we have from (4.8) that

$$\begin{aligned} J'_{12} &\leq CR^2 \int_0^{t/2} e^{-c(t-\tau)} (1+t)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ &\leq CR^2 e^{-ct} \end{aligned}$$

Applying (2.26) with $p = 2, j = k + 2$ and $l = 0$, we get

$$\begin{aligned} J_2 &\leq C \int_{t/2}^t \|\partial_x^{k+2}(f(u) - \beta g(u_t)(\tau))\|_{L^2} d\tau \\ &\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - \beta g(u_t)(\tau))\|_{L^2} \\ &\leq CR^2 \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k+1}{2}} \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR^2 (1+t)^{-\frac{k+1}{2}}. \end{aligned}$$

Thus we have

$$(1+t)^{-\frac{k+1}{2}} \|\partial_x^k \Phi(u)_t\|_{L^2} \leq CE_1 + CR^2. \quad (4.12)$$

Combining (4.9) and (4.12) and taking E_0 and R suitably small, we obtain $\|\Phi(u)\|_X \leq R$.

For $u, \tilde{u} \in X_R$, by using (3.6) we obtain

$$\begin{aligned} \|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} &= \int_0^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\ &= \int_0^{t/2} \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\ &\quad + \int_{t/2}^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\ &=: J_1 + J_2 \end{aligned}$$

By applying (2.25) with $p = 1, j = 0$ and $l = 0$, we have

$$\begin{aligned} J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \|(f(u) - f(\tilde{u})(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\ &=: J_{11} + J_{12} \end{aligned}$$

Using (3.2), we get

$$\begin{aligned}
J_{11} &\leq CR\|u - \tilde{u}\|_X \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} d\tau \\
&\leq CR\|u - \tilde{u}\|_X (1+t)^{-\frac{k}{2}} \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\
&\leq CR\|u - \tilde{u}\|_X (1+t)^{-\frac{k}{2}}
\end{aligned}$$

Also, we have

$$\begin{aligned}
J_{12} &\leq \int_0^{t/2} e^{-c(t-\tau)} \left[(\|\partial_x^k u\|_{L^2} + \|\partial_x^k \tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} \right. \\
&\quad + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^k(u - \tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^\infty} \\
&\quad \left. + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^k(u_t - \tilde{u}_t)\|_{L^2} \right] d\tau \\
&\leq CR\|u - \tilde{u}\|_X \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}} d\tau \\
&\leq CR\|u - \tilde{u}\|_X e^{-ct}.
\end{aligned}$$

To estimate the term J_2 , apply (2.25) with $p = 1, j = k, l = 2$. We obtain

$$\begin{aligned}
J_2 &\leq C \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^1} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau \\
&=: J_{21} + J_{22}
\end{aligned}$$

By using (3.2), we get

$$\begin{aligned}
J_{21} &\leq \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} \left[(\|\partial_x^k u\|_{L^2} + \|\partial_x^k \tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^2} \right. \\
&\quad + (\|u\|_{L^2} + \|\tilde{u}\|_{L^2}) \|\partial_x^k(u - \tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^2} \\
&\quad \left. + (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) \|\partial_x^k(u_t - \tilde{u}_t)\|_{L^2} \right] d\tau \\
&\leq CR\|u - \tilde{u}\|_X \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} (1+\tau)^{-\frac{k}{2}} d\tau \\
&\leq CR\|u - \tilde{u}\|_X (1+t)^{-\frac{k}{2}}
\end{aligned}$$

Finally, we estimate the term J_{22} as

$$\begin{aligned}
J_{22} &\leq \int_{t/2}^t e^{-c(t-\tau)} \left[(\|\partial_x^{k+2} u\|_{L^2} + \|\partial_x^{k+2} \tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^2} \right. \\
&\quad + (\|u\|_{L^2} + \|\tilde{u}\|_{L^2}) \|\partial_x^k(u - \tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^2} \\
&\quad \left. + (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) \|\partial_x^k(u_t - \tilde{u}_t)\|_{L^2} \right] d\tau \\
&\leq CR\|u - \tilde{u}\|_X \int_{t/2}^t e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\
&\leq CR\|u - \tilde{u}\|_X (1+\tau)^{\frac{k}{2}}
\end{aligned}$$

Thus we have shown that

$$(1+t)^{\frac{k}{2}} \|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} \leq CR \|u - \tilde{u}\|_X. \quad (4.13)$$

Suppose that $u, \tilde{u} \in X_R$. It follows from (3.6) that

$$\begin{aligned} \|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} &= \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &= \int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &\quad + \int_{t/2}^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &=: J_1 + J_2. \end{aligned}$$

By using (2.26) with $p = 1, j = 0$, we have

$$\begin{aligned} J_1 &\leq C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|f(u) - f(\tilde{u})(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\ &\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}_t) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &=: J_{11} + J_{12}. \end{aligned}$$

By (3.2), we obtain

$$\begin{aligned} J_{11} &\leq \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} (\|u\|_{L^2} + \|\tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^2} \\ &\quad \times (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) (\|u_t - \tilde{u}_t\|_{L^2}) d\tau \\ &\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{k+1}{2}} \int_0^{t/2} (1+t\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR \|u - \tilde{u}\|_X (1+t)^{-\frac{k+1}{2}}. \end{aligned}$$

For the term J_{12} , by (3.2) we get

$$\begin{aligned} J_{12} &\leq \int_0^{t/2} e^{-c(t-\tau)} [(\|\partial_x^{k+2}u\|_{L^2} + \|\partial_x^{k+2}\tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} \\ &\quad + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u})\|_{L^2} + (\|\partial_x^{k+2}u_t\|_{L^2} + \|\partial_x^{k+2}\tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^\infty} \\ &\quad + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^{k+2}(u_t - \tilde{u}_t)\|_{L^2}] d\tau \\ &\leq CR \|u - \tilde{u}\|_X \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{k+2}{2}} (1+\tau)^{-\frac{n}{4}} d\tau \\ &\leq CR \|u - \tilde{u}\|_X e^{-ct} \end{aligned}$$

By applying (2.26) with $p = 2$, $j = k + 2$ and $l = 0$, we conclude that

$$\begin{aligned}
\hat{J}_2 &\leq C \int_{t/2}^t \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\
&\leq \int_{t/2}^t (\|\partial_x^{k+2}u\|_{L^2} + \|\partial_x^{k+2}\tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u})\|_{L^2} \\
&\quad + (\|\partial_x^{k+2}u_t\|_{L^2} + \|\partial_x^{k+2}\tilde{u}_t\|_{L^2}) \|u - \tilde{u}_t\|_{L^\infty} + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u}_t)\|_{L^2} d\tau \\
&\leq CR \|u - \tilde{u}\|_X \int_{t/2}^t (1 + \tau)^{-\frac{n}{4} - \frac{k+2}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1 + t)^{-\frac{k+1}{2}} \int_{t/2}^t (1 + \tau)^{-\frac{n}{4} - \frac{1}{2}} d\tau \\
&\leq CR \|u - \tilde{u}\|_X (1 + t)^{-\frac{k+1}{2}}.
\end{aligned}$$

Consequently, we have shown that

$$(1 + t)^{\frac{k+1}{2}} \|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))_t\|_X \leq CR \|u - \tilde{u}\|_X. \quad (4.14)$$

Using (4.12) and (4.14) and taking R suitably small, it yields

$$\|\Phi(u) - \Phi(\tilde{u})\|_X \leq \frac{1}{2} \|u - \tilde{u}\|_X. \quad (4.15)$$

From (4.15), we conclude that Φ is a contracting mapping. Then there exists a fixed point $u \in X_R$ of mapping Φ , which is a solution (1.3) and (1.5) and the proof is completed. \square

Finally we study the asymptotic linear profile of the solution.

Suppose that u_L given by the formula $u_L(t) = G(t) * u_1 + H(t) * u_0$. In the previous two section, we have shown that the solution u to the problem (1.3) and (1.5) can be approximated by the linear solution u_L . Now the aim is to derive a simpler asymptotic profile of the linear solution u_L .

In the Fourier space, we obtain $\hat{u}_L(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi, t) + \hat{H}(\xi, t)\hat{u}_0(\xi)$, where $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ are given explicitly in (2.7) and (2.8). First we give the asymptotic expansions of $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ for $\xi \rightarrow 0$. By Using the Taylor expansion to (2.5), we obtain

$$\begin{aligned}
\lambda_{\pm}(\xi) &= \frac{1}{2}(\alpha|\xi|^2 - |\xi|^4) \pm \frac{|\xi|i}{2}(2 + |\xi|^2 - \frac{\alpha^2}{4}|\xi|^4 + O(|\xi|^4)) \\
&= \pm i|\xi| + \frac{\alpha}{2}|\xi|^2 + O(|\xi|^3)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\lambda_+ - \lambda_-} &= \frac{1}{i|\xi|\sqrt{4 + 4|\xi|^2 - |\xi|^6 - \alpha^2|\xi|^4 + 2\alpha|\xi|^5}} \\
&= \frac{1}{2i|\xi|}(1 - \frac{1}{2}|\xi|^2 + O(|\xi|^4)).
\end{aligned}$$

Substituting these expansions to (2.7) and (2.8), we obtain

$$\begin{aligned}\hat{G}(\xi, t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \\ &= \frac{1}{2i|\xi|} \left(e^{\frac{\alpha}{2}|\xi|^2 t} (e^{|\xi|ti} - e^{-|\xi|ti}) + e^{\frac{\alpha}{2}|\xi|^2 t} (O(|\xi|^2) + O(|\xi|^3 t)) \right)\end{aligned}$$

and

$$\begin{aligned}\hat{H}(\xi, t) &= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \\ &= \frac{1}{2} e^{\frac{\alpha}{2}|\xi|^2 t} (e^{|\xi|ti} + e^{-|\xi|ti}) + e^{\frac{\alpha}{2}|\xi|^2 t} (O(|\xi|^2) + O(|\xi|^3 t))\end{aligned}$$

for $\xi \rightarrow 0$. Let

$$\hat{G}_0(\xi, t) = \frac{1}{2i|\xi|} e^{\frac{\alpha}{2}|\xi|^2 t} (e^{i|\xi|t} - e^{-i|\xi|t})$$

and

$$\hat{H}_0(\xi, t) = \frac{1}{2} e^{\frac{\alpha}{2}|\xi|^2 t} (e^{i|\xi|t} + e^{-i|\xi|t}).$$

Thus for $|\xi| \leq r_0$ we obtain

$$|(\hat{G} - \hat{G}_0)(\xi, t)| \leq C e^{-c|\xi|^2 t}, \quad |(\hat{H} - \hat{H}_0)(\xi, t)| \leq C |\xi| e^{-c|\xi|^2 t}$$

where r_0 is a small positive constant. Now we define \bar{u}_L by

$$\bar{u}_L(t) = G_0(t) * u_1 + H_0(t) * u_0. \quad (4.16)$$

\bar{u}_L gives an asymptotic profile of the linear solution u_L .

Theorem 4.2. Suppose that $n \geq 1$, $s \geq 0$ and $u_0 \in H^{s+2} \cap L^1$ and $u_1 \in H^s \cap \dot{W}^{-1,1}$. Put $E_0 = \|u_0\|_{L^1} + \|u_1\|_{\dot{W}^{-1,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}$. Let u_L be the linear solution and \bar{u}_L be defined by (4.16). Thus we have

$$\|\partial_x^k (u_L - \bar{u}_L)(t)\|_{L^2} \leq C E_0 (1+t)^{-\frac{n}{4} - \frac{k+1}{2}} \quad (4.17)$$

for $0 \leq k \leq s+2$.

Proof. It follows from definition that

$$(u - \bar{u}_L)(t) = (G - G_0)(t) * u_1 + (H - H_0)(t) * u_0.$$

So it suffices to show the following estimates:

$$\|\partial_x^k (G - G_0)(t) * u_1\|_{L^2} \leq C (1+t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1-j}{2}} \|\partial_x^j u_1\|_{\dot{W}^{-1,p}} + C e^{-ct} \|\partial_x^{k+l-2} u_1\|_{L^2},$$

$$\|\partial_x^k (H - H_0)(t) * u_0\|_{L^2} \leq C (1+t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1-j}{2}} \|\partial_x^j u_0\|_{L^p} + C e^{-ct} \|\partial_x^{k+l} \phi\|_{L^2},$$

where $1 \leq p \leq 2$, and k, j and l are nonnegative integers such that $0 \leq j \leq k+1$. We assumed $k+l-2 \geq 0$ in the first estimate. These estimates can be proved similarly as in the proof of Lemma 3.2 by using (4.2) for $|\xi| \leq r_0$ and (2.19) and (2.19) and (4.10) for $|\xi| \geq r_0$. We omit the details. \square

References

- [1] H. Aspe, M. C. Depassier, Evolution equation of surface waves in a convecting fluid, *Phys. Rev. A* 41 (1990) 3125–3128.
- [2] R. D. Benguria, M. C. Depassier, Oscillatory instabilities in the Rayleigh-Bénard problem with a free surface, *Phys. Fluids* 30 (1987) 1678–1682.
- [3] R. D. Benguria, M. C. Depassier, On the linear stability theory of Bénard-Marangoni convection, *Phys. Fluids A* 1 (1989) 1123–1127.
- [4] D.J. Benney, Long waves on liquid films, *J. Math. Phys.* 45 (1966) 150–155.
- [5] J. Boussinesq, Théorie des ondes et de remous qui se propagent le long d’un canal rectangulaire horizontal en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *J. Math. Pures Appl.* 217 (1872) 55–108.
- [6] C.I. Christov, Dissipative quasi-particles: the generalized wave equation approach, *Internat. J. Bifur. Chaos* 12 (2002) 2435–2444.
- [7] C.I. Christov, G.A. Maugin, A.V. Porubov, On Boussinesq’s paradigm in nonlinear wave propagation, *C. R. Mec.* 335 (2007) 521–535.
- [8] C.I. Christov, M.G. Velarde, Evolution and interactions of solitary wave (solitons) in nonlinear dissipative systems, *Phys. Scr.* T55 (1994) 101–106.
- [9] C.I. Christov, M.G. Velarde, Dissipative Solitons, *Physica D* 86 (1995) 323–347.
- [10] B. I. Cohen, J. A. Krommes, W. M. Tang, M. N. Rosenbluth, Non-linear saturation of the dissipative trapped-ion mode by mode coupling, *Nucl. Fusion* 16 (1976) 971–992.
- [11] C. Elphick, G.R. Ierley, O. Regev, E.A. Spiegel, Interacting localized structures with Galilean invariance, *Phys. Rev. A* 44 (1991) 1110–1122.
- [12] A. Esfahani, Instability of the stationary solutions of generalized dissipative Boussinesq equation, *Appl. Math.* 59 (2014) 345–358.
- [13] A. N. Garazo, M. G. Velarde, Dissipative Korteweg-de Vries description of Marangoni-Bénard oscillatory convection, *Phys. Fluids A* 3 (1991) 2295–2300.
- [14] A.K. Hobbs, P. Metzener, Dynamical patterns in directional solidification, *Physica D* 93 (1996) 23–51.
- [15] B. Janiaud, A. Pumir, D. Bensimon, V. Croquette, H. Richter, L. Kramer, The Eckhaus instability for travelling waves, *Physica D* 55 (1992) 269–286.
- [16] R. A. Kraenkel, S. M. Kurcbart, J. G. Pereira, M. A. Manna, Dissipative Boussinesq system of equations in the Bénard-Marangoni phenomenon, *Phys. Rev. E* 49 (1994) 1759–1762.
- [17] M. Liu, W. Wang, Global existence and pointwise estimates of solutions for the multidimensional generalized Boussinesq-type equation, *Commun. Pure Appl. Anal.* 13 (2014) 1203–1222.
- [18] J. Mason, E. Knobloch, Long dynamo waves, *Physica D* 205 (2005) 100–124.
- [19] A. A. Nepomnyashchy, M. G. Velarde, A three dimensional description of solitary waves and their interaction in Marangoni-Bénard layers, *Phys. Fluids* 6 (1994) 187–198.

- [20] A. Oron, D.A. Edwards, Stability of a falling liquid film in the presence of interracial viscous stress, *Phys. Fluids* 5 (1993) 506–508.
- [21] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations, *Acta Astronaut.* 4 (1977) 1177–1206.
- [22] J. Topper, T. Kawahara, Approximate equations for long nonlinear waves on a viscous fluid, *J. Phys. Soc. Jpn.* 44 (1978) 663–666.
- [23] V. Varlamov, On the Cauchy problem for the damped Boussinesq equation, *Diff. Int. Equ.* 9 (1996) 619–634.
- [24] V. Varlamov, On spatially periodic solutions of the damped Boussinesq equation, *Diff. Int. Equ.* 10 (1997) 1197–1211.
- [25] V. Varlamov, Long-time asymptotics of solutions of the second initial-boundary value problem for the damped Boussinesq equation, *Abstr. Appl. Anal.* 2 (1998) 281–289.
- [26] V. Varlamov, Eigenfunction expansion method and the long-time asymptotics for the damped Boussinesq equation, *Discrete Contin. Dyn. Syst.* 7 (2001) 675–702.
- [27] V. Varlamov, Two-dimensional Boussinesq equation in a disc and anisotropic Sobolev spaces, *C. R. Mec.* 335 (2007) 548–558.
- [28] V. Varlamov, A. Balogh, Forced nonlinear oscillations of elastic membranes, *Nonlinear Anal. RWA* 7 (2006) 1005–1028.
- [29] Y. Wang, Asymptotic decay estimate of solutions to the generalized damped Bq equation, *J. Inequal. Appl.* 2013 (2013) 323, 12 pp.
- [30] Y. Wang, On the Cauchy problem for one dimension generalized Boussinesq equation, *Internat. J. Math.* 26 (2015) 1550023, 22 pp.
- [31] T. Yamada, Y. Kuramoto, A reduced model showing chemical turbulence, *Prog. Theor. Phys.* 56 (1976) 681–683.
- [32] S. M. Zheng, *Nonlinear Evolution Equations, Monographs and Surveys in Pure and Applied Mathematics*, 133, Chapman Hall/CRC, 2004.